Oxford Cambridge and RSA

Tuesday 7 June 2022 – Afternoon

A Level Mathematics B (MEI)

H640/01 Pure Mathematics and Mechanics

Time allowed: 2 hours

* 9 4 6 5 4 6 5 2 8 8 *

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where $ {}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient Rule $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$ **or** $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \sum(x_i - \bar{x})^2 = \sum x_i^2 - \dfrac{(\sum x_i)^2}{n} = \sum x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = r) = {}^nC_r\, p^r q^{n-r}$ where $q = 1 - p$

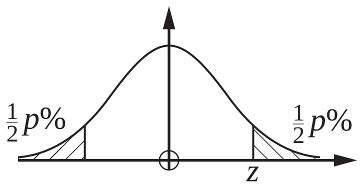
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

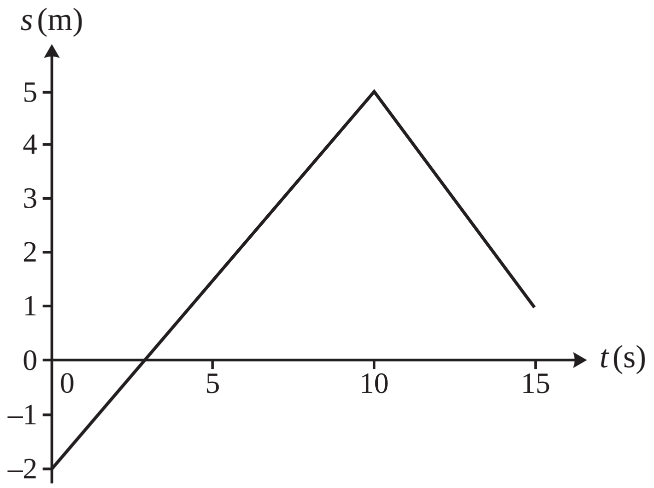
$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (24 marks)

1 A particle moves along a straight line. The displacement s m at time t s is shown in the displacement-time graph below. The graph consists of straight line segments joining the points $(0, -2)$, $(10, 5)$ and $(15, 1)$.



(a) Find the distance travelled by the particle in the first 15 s. **[2]**

(b) Calculate the velocity of the particle between $t = 10$ and $t = 15$. **[2]**

2 Express $\frac{13-x}{(x-3)(x+2)}$ in partial fractions. **[3]**

3 **(a)** Sketch the graph of $y = \arctan x$ where x is in radians. **[2]**

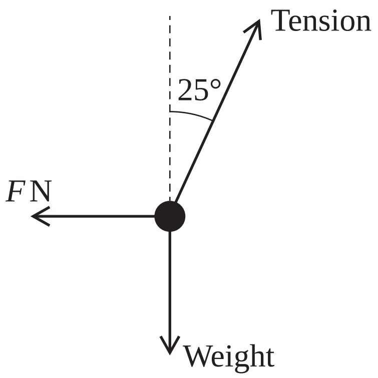
(b) **In this question you must show detailed reasoning.**

Find all points of intersection of the curves $y = 3\sin x \cos x$ and $y = \cos^2 x$ for $-\pi \leqslant x \leqslant \pi$. **[6]**

4 Using an appropriate expansion show that, for sufficiently small values of x,

$\frac{1-x}{(2+x)^2} \approx \frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2$. **[4]**

5 A sphere of mass 3 kg hangs on a string. A horizontal force of magnitude F N acts on the sphere so that it hangs in equilibrium with the string making an angle of 25° to the vertical. The force diagram for the sphere is shown below.



(a) Sketch the triangle of forces for these forces. **[2]**

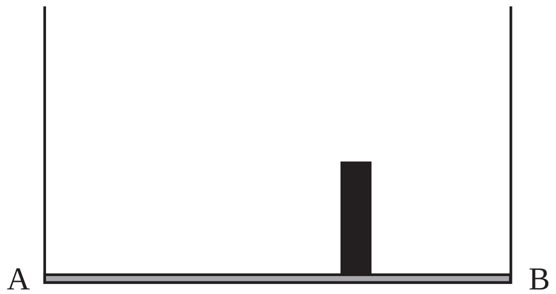
(b) Hence or otherwise determine each of the following:

- the tension in the string
- the value of F. **[3]**

Answer **all** the questions.

Section B (76 marks)

6 A shelf consists of a horizontal uniform plank AB of length 0.8 m and mass 5 kg with light inextensible vertical strings attached at each end. A stack of bricks each of mass 2.3 kg is placed on the plank as shown in the diagram.



(a) Explain the meaning of each of the following modelling assumptions.

- The stack of bricks is modelled as a particle.
- The plank is modelled as uniform. **[2]**

Either of the strings will break if the tension exceeds 75 N.

(b) Find the greatest number of bricks that can be placed at the centre of the plank without breaking the strings. **[2]**

(c) Find an expression for the moment about A of the weight of a stack of n bricks when the stack is at a distance of x m from A. State the units for your answer. **[2]**

(d) Calculate the greatest distance from A that the largest stack of bricks can be placed without a string breaking. **[3]**

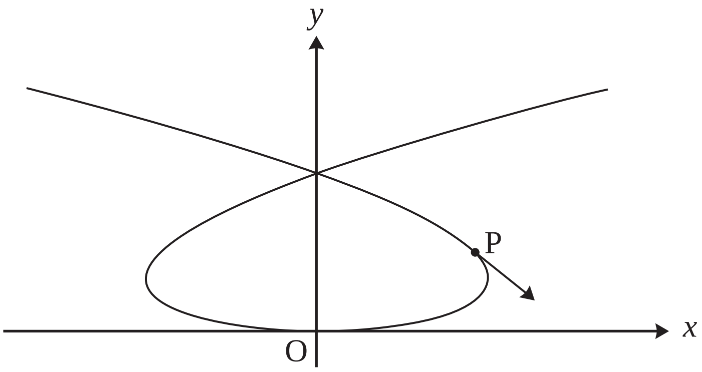
7 In this question the x- and y-directions are horizontal and vertically upwards respectively and the origin is on horizontal ground.

A ball is thrown from a point 5 m above the origin with an initial velocity $\begin{pmatrix} 14 \\ 7 \end{pmatrix}$ m s^{-1}.

(a) Find the position vector of the ball at time t s after it is thrown. **[3]**

(b) Find the distance between the origin and the point at which the ball lands on the ground. **[3]**

8 A particle moves in the x-y plane so that its position at time t s is given by $x = t^3 - 8t,\ y = t^2$ for $-3.5 < t < 3.5$. The units of distance are metres. The graph shows the path of the particle and the direction of travel at the point P (8, 4).



(a) Find $\frac{dy}{dx}$ in terms of t. **[3]**

(b) Hence show that the value of $\frac{dy}{dx}$ at P is -1. **[2]**

(c) Find the time at which the particle is travelling in the direction opposite to that at P. **[2]**

(d) Find the cartesian equation of the path, giving x^2 as a function of y. **[3]**

9 In this question, the vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

The velocity $\mathbf{v}\,\text{m s}^{-1}$ of a particle at time t s is given by $\mathbf{v} = kt^2\mathbf{i} + 6t\mathbf{j}$, where k is a positive constant. The magnitude of the acceleration when $t = 2$ is $10\,\text{m s}^{-2}$.

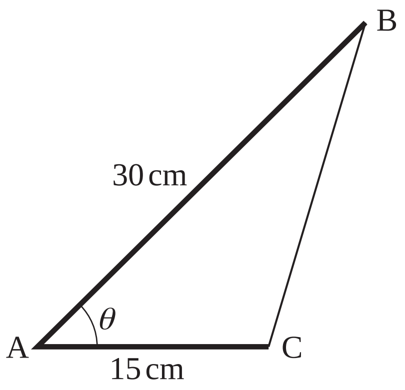
(a) Calculate the value of k. **[4]**

The particle is at the origin when $t = 0$.

(b) Determine an expression for the position vector of the particle at time t. **[2]**

(c) Determine the time when the particle is directly north-east of the origin. **[2]**

10 A triangle ABC is made from two thin rods hinged together at A and a piece of elastic which joins B and C. AB is a 30 cm rod and AC is a 15 cm rod. The angle BAC is θ radians as shown in the diagram.



The angle θ increases at a rate of 0.1 radians per second.

Determine the rate of change of the length BC when $\theta = \frac{1}{3}\pi$. **[8]**

11 Given that k is a positive constant, show that $\int_k^{2k} \frac{2}{(2x+k)^2}\,dx$ is inversely proportional to k. **[6]**

12 Prove by contradiction that 3 is the only prime number which is 1 less than a square number. **[4]**

13 A toy train consists of an engine of mass 0.5 kg pulling a coach of mass 0.4 kg. The coupling between the engine and the coach is light and inextensible. The train is pulled along with a string attached to the front of the engine.

At first, the train is pulled from rest along a horizontal carpet where there is a resistance to motion of 0.8 N on each part of the train. The string is horizontal, and the tension in the string is 5 N.

(a) Determine the velocity of the train after 1.5 s. **[4]**

The train is then pulled up a track inclined at 20° to the horizontal. The string is parallel to the track and the tension in the string is P N. The resistance on each part of the train along the track is R N.

(b) Draw a diagram showing all the forces acting on the train modelled as two connected particles. **[3]**

(c) Find the equation of motion for the train modelled as a single particle. **[2]**

(d) The acceleration of the train when $P = 5.5$ is double the acceleration when $P = 5$.

Calculate the value of R. **[3]**

14 Alex places a hot object into iced water and records the temperature θ °C of the object every minute. The temperature of an object t minutes after being placed in iced water is modelled by $\theta = \theta_0 e^{-kt}$ where θ_0 and k are constants whose values depend on the characteristics of the object.

The temperature of Alex's object is 82 °C when it is placed into the water. After 5 minutes the temperature is 27 °C.

(a) Find the values of θ_0 and k that best model the data. **[3]**

(b) Explain why the model may **not** be suitable in the long term if Alex does not top up the ice in the water. **[1]**

(c) Show that the model with the values found in part **(a)** can be written as $\ln\theta = a - bt$ where a and b are constants to be determined. **[2]**

Ben places a different object into iced water at the same time as Alex. The model for Ben's object is $\ln\theta = 3.4 - 0.08t$.

(d) Determine each of the following:

- the initial temperature of Ben's object
- the rate at which Ben's object is cooling initially. **[4]**

(e) According to the models, there is a time at which both objects have the same temperature.

Find this time and the corresponding temperature. **[3]**

END OF QUESTION PAPER

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